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Harmonic Univalent Functions with Negative Coefficients

Herb Silverman*

Department of Mathematics, College of Charleston, Charleston, South Carolina 29424

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We give sufficient coefficient conditions for normalized harmonic functions to map onto either starlike or convex regions. These conditions are also shown to be necessary when the coefficients are negative. This leads to distortion bounds and extreme points.

1. INTRODUCTION

A continuous function $f = u + iv$ is a complex-valued harmonic function in a domain $\mathscr{D} \subset C$ if both u and v are real harmonic in $\mathscr{D}$. In any simply connected domain we can write

$$f = h + \bar{g}, \tag{1}$$

where h and g are analytic in $\mathscr{D}$. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in $\mathscr{D}$ is that $|h'(z)| > |g'(z)|$ in $\mathscr{D}$. See [2].

Denote by S_H the class of functions f of the form (1) that are harmonic univalent and sense-preserving in the unit disk $\Delta = \{z : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$ and by S_H^0 the subclass of S_H for which $F_{\bar{z}}(0) = 0$. Observe that S_H reduces to S, the class of normalized univalent analytic functions, if the co-analytic part of f is zero.

In this note, we will look at various subclasses of S_H^0, so that h and g may be expressed as

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=2}^{\infty} b_n z^n. \tag{2}$$

* This work was completed while the author was on sabbatical leave as a Visiting Scholar at the University of California at San Diego. E-mail: silvermanh@cofc.edu.

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Let S_H^{*0} and K_H^0 be the subclasses of S_H^0 consisting of functions f that map Δ onto starlike and convex domains, respectively. We further denote by T_H^{*0} and TK_H^0 the subclasses of S_H^{*0} and K_H^0, respectively, whose coefficients $g = h + \bar{g}$ take the form

$$h(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0; \qquad g(z) = -\sum_{n=2}^{\infty} b_n z^n, \quad b_n \geq 0. \quad (3)$$

Many of the standard results for S and its subclasses have no harmonic analog. For example, some functions in S_H have growth larger than $0(1/(1-r)^2)$ $(r \to 1)$, $\int_0^z (f(t)/t)\,dt$ may be in K_H^0 even though f is not even univalent, the Caratheodory extension theorem does not hold, and the hereditary property for convex conformal maps fails in the harmonic case. See [2].

We will investigate comparisons between harmonic functions $f = h + \bar{g}$ whose coefficients are of the form (3) and their analytic counterpart. See [4]. In the sequel, we shall assume that $f = h + \bar{g}$, where h and g are either of the form (2) or (3).

2. MAIN RESULTS

For $f = h + \bar{g}$ of the form (2), it was shown in [3] that $|a_n| \leq (n+1)(2n+1)/6$, $|b_n| \leq (n-1)(2n-1)/6$ for $f \in S_H^{*0}$. It was shown in [2] that $|a_n| \leq (n+1)/2$, $|b_n| \leq (n-1)/2$ for $f \in K_H^0$. All results are sharp. We thus have necessary coefficient conditions for these classes. We now give sufficient conditions, which we will show are also necessary when f is of the form (3).

THEOREM 1. *If f of the form* (2) *satisfies* $\sum_{n=2}^{\infty} n(|a_n| + |b_n|) \leq 1$, *then* $f \in S_H^{*0}$.

Proof. Note first that

$$|h'(z)| \geq 1 - \sum_{n=1}^{\infty} n|a_n| r^{n-1} > \sum_{n=2}^{\infty} n|b_n| r^{n-1} \geq |g'(z)|,$$

so that f is locally univalent and sense-preserving. It suffices to show that $\partial/\partial\theta(\arg f(re^{i\theta})) > 0$, $0 \leq \theta < 2\pi$, $0 < r < 1$. We have

$$f(re^{i\theta}) = re^{i\theta} + \sum_{n=2}^{\infty} \left(a_n e^{in\theta} + \bar{b}_n e^{-in\theta}\right) r^n,$$

and

$$\frac{\partial}{\partial\theta}\left(\arg f(re^{i\theta})\right) = lm\left(\frac{\partial}{\partial\theta}\log f(re^{i\theta})\right)$$
$$= \operatorname{Re}\left\{\frac{1+\sum_{n=2}^{\infty} n\left(a_n e^{i(n-1)\theta} - \bar{b}_n e^{-i(n+1)\theta}\right) r^{n-1}}{1+\sum_{n=2}^{\infty}\left(a_n e^{i(n-1)\theta} + \bar{b}_n e^{-i(n+1)\theta}\right) r^{n-1}}\right\}$$
$$:= \operatorname{Re}\frac{1+A(z)}{1+B(z)}.$$

Setting

$$\frac{1+A(z)}{1+B(z)} = \frac{1+w(z)}{1-w(z)}, \tag{4}$$

we will have $\partial/\partial\theta \arg f(re^{i\theta}) > 0$ if $|w(z)| \le r$. But

$$w(z) = \frac{A-B}{2+A+B}$$
$$= \frac{\sum_{n=2}^{\infty}\left[(n-1)a_n e^{i(n-1)\theta} - (n+1)\bar{b}_n e^{-i(n+1)\theta}\right] r^{n-1}}{2+\sum_{n=2}^{\infty}\left[(n+1)a_n e^{i(n-1)\theta} - (n-1)\bar{b}_n e^{-i(n+1)\theta}\right] r^{n-1}},$$

so that

$$|w(z)| \le \frac{\sum_{n=2}^{\infty}\left[(n-1)|a_n| + (n+1)|b_n|\right] r}{2-\sum_{n=2}^{\infty}\left[(n+1)|a_n| + (n-1)|b_n|\right] r}.$$

This last expression is bounded above by r if and only if $\sum_{n=2}^{\infty} n(|a_n| + |b_n|) \le 1$.

COROLLARY 1. *If f of the form* (2) *satisfies* $\sum_{n=2}^{\infty} n^2(|a_n| + |b_n|) \le 1$, *then* $f \in K_H^0$.

Proof. For $f \in S_H^{*0}$ it is shown in [1] that $\int_0^z (f(\zeta)/\zeta)\, d\zeta \in K_H^0$, where integration is along the line segment from 0 to z. The result now follows from Theorem 1.

Unlike in the analytic case, a function harmonic starlike (or even harmonic convex) when $|z| = R$ need not be univalent for $|z| < R$. However, the coefficient bounds of Theorem 1 do provide us with a sufficient condition for univalence.

COROLLARY 2. *Under the conditions of Theorem* 1, *f is also harmonic univalent in* Δ.

Proof. If $g(z) \equiv 0$, then $f(z)$ is analytic and the univalence of f follows from its starlikeness. If $g(z) \not\equiv 0$ and $z_1 \neq z_2$, then

$$\begin{aligned} \left| \frac{f(z_1) - f(z_2)}{h(z_1) - h(z_2)} \right| &\geq 1 - \left| \frac{g(z_1) - g(z_2)}{h(z_1) - h(z_2)} \right| \\ &= 1 - \left| \frac{\sum_{n=2}^{\infty} b_n (z_1^n - z_2^n)}{(z_1 - z_2) + \sum_{n=2}^{\infty} a_n (z_1^n - z_2^n)} \right| \\ &> 1 - \frac{\sum_{n=2}^{\infty} n|b_n|}{1 - \sum_{n=2}^{\infty} n|a_n|} \geq 0, \end{aligned}$$

which proves univalence.

Remark. The restrictions in Theorem 1 placed on the moduli of coefficients enabled us to conclude for arbitrary rotations of the coefficients of f that the resulting functions would still be harmonic starlike and univalent. Our next theorem establishes that such coefficient bounds cannot be improved.

THEOREM 2. *For f of the form* (3), $f \in T_H^{*0}$ *if and only if* $\sum_{n=2}^{\infty} n(a_n + b_n) \leq 1$.

Proof. In view of Theorem 1, we need only show that $f \notin T_H^{*0}$ if the coefficient condition does not hold. For this case, we will show that f is not even univalent. Setting $z = r > 0$, we have $f(r) = r - \sum_{n=2}^{\infty} (a_n + b_n) r^n$ and $f'(r) = 1 - \sum_{n=2}^{\infty} n(a_n + a_n) r^{n-1}$. Since $f'(0) = 1$ and $f'(1) < 0$, there must exist an $r_0, r_0 < 1$, for which $f'(r_0) = 0$. Hence, r_0 is a local max for $f(r)$ and $f(r)$ is not 1-1 on the real interval $(0, 1)$.

COROLLARY 1. *For f of the form* (3), $f \in T_H^{*0}$ *if and only if f is harmonic univalent.*

COROLLARY 2. *If $f \in T_H^{*0}$, then $r - r^2/2 \leq |f(z)| \leq r + r^2/2$ ($|z| = r$). The result is sharp, with equality for $z - z^2/2$ and $z - \bar{z}^2/2$.*

Proof. Noting that $2\sum_{n=2}^{\infty} (a_n + b_n) \leq \sum_{n=2}^{\infty} n(a_n + b_n) \leq 1$, we have

$$|f(z)| \leq r + \sum_{n=2}^{\infty} (a_n + b_n) r^n \leq r + r^2 \sum_{n=2}^{\infty} (a_n + b_n) \leq r + r^2/2$$

and

$$|f(z)| \geq r - \sum_{n=2}^{\infty} (a_n + b_n) r^n \geq r - r^2 \sum_{n=2}^{\infty} (a_n + b_n) \geq r - r^2/2.$$

COROLLARY 3. *If* $f \in T_H^{*0}$, *then* $\{w : |w| < 1/2\} \subset f(\Delta)$.

Proof. This covering result follows from the left hand inequality in Corollary 2.

Remark. If the co-analytic part of f in Theorems 1 and 2 is zero, the results reduce to sufficient coefficient conditions for starlike analytic functions. See [4]. There is, however, an interesting difference. In the analytic case, we also have

$$\left|\frac{\partial}{\partial\theta}\arg f(re^{i\theta}) - 1\right| = \left|\frac{zf'(z)}{f(z)} - 1\right| < 1, \qquad z \in \Delta,$$

so that the values of zf'/f lie in the subset of the right-half plane consisting of the disk centered at 1 and having radius 1. For the harmonic case with f of the form (2),

$$\frac{\partial}{\partial\theta}\arg f(re^{i\theta}) - 1 = \frac{\sum_{n=2}^{\infty}\left[(n-1)a_n e^{i(n-1)\theta} - (n+1)\bar{b}_n e^{-i(n+1)\theta}\right]r^{n-1}}{1 + \sum_{n=2}^{\infty}\left(a_n e^{i(n-1)\theta} + \bar{b}_n e^{-i(n+1)\theta}\right)r^{n-1}}.$$

When $f(z) = z - \bar{z}^2/2$, we get $|\partial/\partial\theta \arg f(re^{i\theta}) - 1| = |3r/2/(1 - e^{-3i\theta}r/2)|$. Setting $\theta = 0$ and letting $r \to 1$, we see that this last expression can be as large as 3.

THEOREM 3. *For f of the form* (3), $f \in TK_H^0$ *if and only if* $\sum_{n=2}^{\infty} n^2(a_n + b_n) \le 1$.

Proof. In view of Corollary 1 to Theorem 1, we need only show that $f \notin TK_H^0$ if the coefficient inequality does not hold. A necessary and sufficient condition for f to map $|z| = r$ onto a convex domain is that

$$\frac{\partial}{\partial\theta}\left(\arg\left\{\frac{\partial}{\partial\theta}f(re^{i\theta})\right\}\right) = \operatorname{Re}\left\{\frac{1 - \sum_{n=2}^{\infty} n^2\left[a_n e^{i(n-1)\theta} + b_n e^{-i(n+1)\theta}\right]r^{n-1}}{1 - \sum_{n=2}^{\infty} n\left[a_n e^{i(n-1)\theta} - b_n e^{-i(n+1)\theta}\right]r^{n-1}}\right\} > 0.$$

If we set $\theta = 0$, the last expression is negative for $\sum_{n=2}^{\infty} n^2(a_n + b_n) > 1$ and r sufficiently close to 1. Hence $f \notin TK_H^0$, and the proof is complete.

COROLLARY. *If* $f \in T_H^{*0}$, *then f maps* $|z| < 1/2$ *onto a convex domain. The result is sharp, with extremal functions* $z - z^2/2$ *and* $z - \bar{z}^2/2$.

Proof. It suffices to show that $2f(z/2) \in TK_H^0$. We have

$$2f(z/2) = z - \sum_{n=2}^{\infty} \frac{a_n}{2^{n-1}} z^n - \sum_{n=2}^{\infty} \frac{b_n}{2^{n-1}} \bar{z}^n$$

and

$$\sum_{n=2}^{\infty} n^2 \left(\frac{a_n}{2^{n-1}} + \frac{b_n}{2^{n-1}} \right) \leq \sum_{n=2}^{\infty} n(a_n + b_n) \leq 1.$$

3. EXTREME POINTS

For any compact family, the maximum or minimum of the real part of any continuous linear functional occurs at one of the extreme points of the closed convex hull. Since both T_H^{*0} and TK_H^0 are convex families, we will use the necessary and sufficient coefficient inequalities of Theorems 2 and 3 to determine their extreme points.

THEOREM 4. (a) *Set* $h_1(z) = z$, $h_n(z) = z - z^n/n$ $(n = 2, 3, \dots)$, *and* $g_n(z) = z - \bar{z}^n$, $(n = 2, 3, \dots)$. *Then* $f \in T_H^{*0}$ *if and only if it can be expressed in the form* $f(z) = \sum_{n=1}^{\infty}(\lambda_n h_n + \gamma_n g_n)$, *where* $\lambda_n \geq 0$, $\gamma_n \geq 0$, $\lambda_1 = 1 - \sum_{n=2}^{\infty}(\lambda_n + \gamma_n) \geq 0$, *and* $\gamma_1 = 0$. *In particular, the extreme points of* T_H^{*0} *are* $\{h_n\}$ *and* $\{g_n\}$.

(b) *Set* $h_1(z) = z$, $h_n(z) = z - z^n/n^2$ $(n = 2, 3, \dots)$, *and* $g_n(z) = z - \bar{z}^n/n^2$ $(n = 2, 3, \dots)$. *Then* $f \in TK_H^0$ *if and only if it can be expressed as* $f(z) = \sum_{n=1}^{\infty}(\lambda_n h_n + \gamma_n g_n)$, *where* $\lambda_n \geq 0$, $\gamma_n \geq 0$, $\lambda_1 = 1 - \sum_{n=2}^{\infty}(\lambda_n + \gamma_n)$, *and* $\gamma_1 = 0$. *The extreme points of* TK_H^0 *and* $\{h_n\}$ *and* $\{g_n\}$.

Proof. We prove only (a). The proof of (b) follows from Theorem 3 as that of (a) follows from Theorem 2. Suppose

$$f(z) = \sum_{n=1}^{\infty} (\lambda_n h_n + \gamma_n g_n) = z - \sum_{n=2}^{\infty} \left(\frac{\lambda_n}{n} z^n + \frac{\gamma_n}{n} \bar{z}^n \right).$$

Then

$$\sum_{n=2}^{\infty} n \left(\frac{\lambda_n}{n} + \frac{\gamma_n}{n} \right) = \sum_{n=2}^{\infty} (\lambda_n + \gamma_n) = 1 - \lambda_1 \leq 1,$$

and $f \in T_H^{*0}$.

Conversely, if $f \in T_H^{*0}$, then $a_n \leq 1/n$ and $b_n \leq 1/n$. Set $\lambda_n = na_n$, $\gamma_n = nb_n$ $(n = 2, 3, \dots)$, $\lambda_1 = 1 - \sum_{n=2}^{\infty} \lambda_n$, and $\gamma_1 = 0$. Then $f(z) = \sum_{n=1}^{\infty}(\lambda_n h_n + \gamma_n g_n)$.

4. POSITIVE ORDER

We say that f of the form (2) is harmonic starlike of order $\alpha, 0 \leq \alpha < 1$, for $|z| = r$ if $\partial/\partial\theta(\arg f(re^{i\theta})) \geq \alpha, |z| = r$. Denote by $S_H^{*0}(\alpha)$ and $T_H^{*0}(\alpha)$ the subclasses of S_H^{*0} and T_H^{*0}, respectively, that are starlike of order α. Many of our results can be rewritten for functions of positive order. For instance, if we replace (4) with

$$\frac{1 + A(z)}{1 + B(z)} = \frac{1 + (1 - 2\alpha)w(z)}{1 - w(z)}, \qquad 0 \leq \alpha < 1,$$

arguments similar to those already given lead to the following result.

THEOREM 5. *A sufficient condition for a function f of the form* (2) *to be in* $S_H^{*0}(\alpha)$ *is that* $\sum_{n=2}^{\infty}(n - \alpha/1 - \alpha)(|a_n| + |b_n|) \leq 1$. *For a function of the form* (3), *this condition is also necessary.*

As in Theorem 4, these necessary and sufficient coefficient conditions for $T_H^{*0}(\alpha)$ lead to the extreme points, namely $h_1(z) = z$, $h_n(z) = z - ((1 - \alpha)/(n - \alpha))z^n$ $(n = 2, 3, \ldots)$, and $g_n(z) = z - ((1 - \alpha)/(n - \alpha))\bar{z}^n$ $(n = 2, 3, \ldots)$. Finally, we remark that the corresponding definition for harmonic convex functions of order α leads to analogous coefficient bounds and extreme points.

REFERENCES

1. J. A. Cima and A. E. Livingston, Integral smoothness properties of some harmonic mappings, *Complex Variables* **11** (1989), 95–110.
2. J. Clunie and T. Sheil-Small, Harmonic univalent functions, *Ann. Acad. Sci. Fenn. Ser. A I Math.* **9** (1984), 3–25.
3. T. Sheil-Small, Constants for planar harmonic mappings, *J. London Math. Soc.* (2) **42** (1990), 237–248.
4. H. Silverman, Univalent functions with negative coefficients, *Proc. Amer. Math. Soc.* **51** (1975), 109–116.